Dynamic scaling in the vicinity of the Luttinger liquid fixed point

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We calculate the single-particle spectral function $A(k,\omega)$ of a one-dimensional Luttinger liquid by means of a functional renormalization group (RG) approach. Given an infrared energy cutoff $\Lambda = \Lambda_0 e^{-l}$, our approach yields the spectral function in the scaling form, $A_\Lambda(k_F + p, \omega) = \tau Z_l \tilde{A}_l(p\xi, \omega\tau)$, where k_F is the Fermi momentum, Z_l is the wave-function renormalization factor, $\tau = 1/\Lambda$ is the time scale and $\xi = v_F/\Lambda$ is the length scale associated with Λ. At the Luttinger liquid fixed point ($l \to \infty$) our RG result for $A(k,\omega)$ exhibits the correct anomalous scaling properties, and for $k = \pm k_F$ agrees exactly with the well-known bosonization result at weak coupling. Our calculation demonstrates that the field rescaling is essential for obtaining the crossover from Fermi liquid behavior to Luttinger liquid behavior from a truncation of the hierarchy of exact RG flow equations as the infrared cutoff Λ is reduced.

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I. INTRODUCTION

For many years the renormalization group (RG) has been used to study interacting Fermi systems in one spatial dimension ($1d$).[1] The success of RG methods in $1d$ relies on the fact that at low energies, the two-body interactions between fermions can be parameterized in terms of four coupling constants, which are usually called g_1 (backward scattering), g_2 (forward scattering of fermions propagating in opposite directions), g_3 (Umklapp scattering), and g_4 (forward scattering of fermions propagating in the same direction). Most authors have focused on the calculation of the RG β-functions, which in the Wilsonian RG[2,3,4] describe the flow of these couplings as the degrees of freedom are eliminated and rescaled.[5] However, the RG β-functions do not completely describe the physical behavior of the system. In particular, the RG β-functions do not contain information about the single-particle excitations. To investigate these, one should calculate the momentum- and frequency-dependent single-particle spectral function $A(k,\omega)$, which at temperature $T = 0$ is related to the imaginary part of the single-particle Green function $G(k,\omega)$ via

$$A(k,\omega) = -\frac{1}{\pi} \operatorname{Im} G(k, \omega + i0) . \tag{1.1}$$

In the absence of backward and Umklapp scattering, the leading asymptotic long-distance and long-time behavior of the single-particle Green function $G(x,t)$ in the space-time domain can be calculated via bosonization if the energy dispersion is linearized around the Fermi points (Tomonaga-Luttinger model, TLM).[6,7] Obviously, in order to obtain the spectral function, one should calculate the Fourier-transform $G(k,\omega)$ of $G(x,t)$, which in general cannot be done exactly. For the spinless TLM with g_2-interactions a mathematically non-rigorous but physically reasonable asymptotic analysis yields at temperature $T = 0$ for wave-vectors close to $\pm k_F$ and for low energies,[8,9,10,11]

$$\begin{aligned} A_{\rm TL}(\pm(k_F+p),\omega) \;&\approx\; \Lambda_0^{-\eta} \frac{\eta}{2} \Theta(\omega^2 - (v_c p)^2) \\ &\times |\omega - v_c p|^{-1+\eta/2} |\omega + v_c p|^{\eta/2} , \end{aligned} \tag{1.2}$$

where Λ_0 is some ultraviolet cutoff with units of energy (for example a band width cutoff). At weak coupling the anomalous dimension η and the velocity v_c of collective charge excitations are to leading order

$$\eta \;\approx\; \frac{\tilde{g}^2}{8} , \tag{1.3}$$

$$v_c \;\approx\; v_F \left(1 - \frac{\tilde{g}^2}{8} \right) . \tag{1.4}$$

Here v_F is the bare Fermi velocity and

$$\tilde{g} = g_2/(\pi v_F) \tag{1.5}$$

is the dimensionless coupling describing forward scattering of electrons propagating in different directions. Eq. (1.2) satisfies a simple scaling law: for an *arbitrary* length ξ we may write

$$A_{\rm TL}(\pm(k_F+p),\omega) = \tau \left(\frac{\xi_0}{\xi}\right)^{\eta} \tilde{A}_{\rm TL}(p\xi, \omega\tau) , \tag{1.6}$$

where $\tau = \xi/v_F$ is the time scale associated with ξ, and the length $\xi_0 = v_F/\Lambda_0$ corresponds to the ultraviolet cutoff Λ_0. The dimensionless scaling function $\tilde{A}_{\rm TL}(q,\epsilon)$ is

$$\tilde{A}_{\rm TL}(q,\epsilon) = \frac{\eta}{2} \Theta(\epsilon^2 - (\tilde{v}q)^2) |\epsilon - \tilde{v}q|^{-1+\eta/2} |\epsilon + \tilde{v}q|^{\eta/2} , \tag{1.7}$$

where q and ϵ are dimensionless variables, and $\tilde{v} = v_c/v_F$ is the dimensionless velocity renormalization factor. Obviously, the dynamic exponent (defined via $\tau \propto \xi^z$) is $z = 1$. The dynamic scaling function (1.7) is scale invariant, so that for an arbitrary scale factor s

$$\tilde{A}_{\rm TL}(sq, s\epsilon) = s^{-1+\eta} \tilde{A}_{\rm TL}(q,\epsilon) . \tag{1.8}$$

This scale invariance is a consequence of the fact that the TLM represents a critical system, corresponding to the Luttinger liquid fixed point.[12] Hence, the RG β-function of the TLM vanishes identically. It is generally accepted that the TLM describes the generic low-energy and long-wavelength properties of one-dimensional Fermi systems with dominant forward scattering. Thus, the TLM is an effective model which in the regime where backward and Umklapp scattering are irrelevant emerges at low energies when the high-energy degrees of freedom are integrated out in a Wilsonian RG.

In the theory of dynamic critical phenomena[13,14] it is usually assumed that correlation functions close to the critical point can be written in the dynamic scaling form (1.6), with some dynamic scaling function $\tilde{A}(p\xi, \omega\tau)$. The dynamic scaling function depends on ξ only implicitly via $p\xi$ and $\omega\tau = \omega\xi/v_F$. For example, at finite temperatures, where we may identify $\xi = v_F/T$ and $\tau \approx 1/T$, the scaling function $\tilde{A}_{\rm TL}(p\xi, \omega\tau)$ of the TLM has been discussed by Orgad, Kivelson, and collaborators.[15]

Suppose now that we iterate the RG to some large scale factor $l = \ln(\Lambda_0/\Lambda)$, and write the resulting spectral function in the scaling form analogous to Eq. (1.6),

$$A_\Lambda(k_F + p, \omega) = \tau \left(\frac{\xi_0}{\xi}\right)^\eta \tilde{A}_l(p\xi, \omega\tau) \ . \tag{1.9}$$

According to the above argument, we would expect that for $l \to \infty$ the scaling function $\tilde{A}_l(q, \epsilon)$ becomes independent of l and approaches $\tilde{A}_{\rm TL}(q, \epsilon)$ for large $|q|$ and $|\epsilon|$. In this work we shall show that this expectation is not quite correct: even if both $|q|$ and $|\epsilon|$ are large compared with unity, the difference $|\epsilon| - \tilde{v}|q|$ can be small. In this regime we find that $\tilde{A}_\infty(q, \epsilon) = \lim_{l\to\infty} \tilde{A}_l(q, \epsilon)$ behaves quite differently from the scaling function of the TLM. Physically, this is due to the fact that the spectral line-shape of a generic one-dimensional Fermi system with dominant forward scattering exhibits in general some non-universal features which are determined by irrelevant couplings and thus spoil the universality of the spectral function. In fact, even the spectral function of the TLM exhibits some non-universal features.[11]

We shall here attempt to calculate the spectral function of the g_2-TLM using RG methods. Therefore we shall calculate the RG flow of the irreducible *two-point* vertex. Recall that the usual RG β-function describes the flow of the momentum- and frequency-independent part of the *four-point* vertex. Surprisingly, the problem of calculating the spectral function of a strongly correlated fermionic system like the TLM via RG methods has not received much attention. In fact, with the exception of the recent work by Ferraz,[16] where the spectral function of a special $2d$ Fermi system has been calculated by means of the field theory RG, we are not aware of any RG calculation of the spectral line shape of a strongly correlated fermionic many-body system. Note that in order to obtain a non-trivial k- and ω-dependence and the anomalous scaling properties given in Eq. (1.6), one should retain infinitely many couplings which are irrelevant by naive power counting. In other words, one needs to keep track of the RG flow of coupling functions. At the first sight, this seems to be a formidable task, which is impossible to carry out in practice. Nevertheless, in this work we shall show that at weak coupling a simple truncation of the hierarchy of functional RG equations for the irreducible vertex functions yields an expression for the spectral function which has the correct anomalous scaling properties and, at least for $k = \pm k_F$, agrees with the exact result known from bosonization.

II. EXACT FLOW EQUATIONS

Originally, exact RG flow equations have been developed in field theory and statistical physics to study systems in the vicinity of a critical point.[17,18,19,20,21] Recently, several authors have started to apply these methods to interacting Fermi systems at finite densities.[22,23,24] In the normal state the existence of a Fermi surface adds some new complexity to the problem: because the single particle Green function exhibits singularities on the entire Fermi surface, the Fermi surface plays the role of a continuum (in $d > 1$) of RG fixed points. However, this fixed point manifold is not known a priori, and should be calculated self-consistently within the RG. In Ref. 25 we have shown how the true Fermi surface of the interacting system can be calculated from the requirement that the RG approaches a fixed point. Fortunately, in $1d$ we know a priori that the Fermi surface (which consists of two points $\pm k_F$) is not renormalized by the interactions,[26] so that the above mentioned problem of a self-consistent determination of the Fermi surface does not arise.

Even if the shape of the Fermi surface is known, the implementation of the exact RG for Fermi systems is non-trivial and some open problems remain. In particular, in Ref. 25 we have pointed out that the RG approach proposed in Refs. 22,23,24 includes only the mode-elimination step, but completely omits the usual rescaling. We have further speculated that this incomplete RG transformation might be responsible for the runaway flow to strong coupling which is typically encountered in these works. The purpose of this article is to further substantiate this claim by applying the functional RG method to a non-trivial yet exactly solvable many-body problem where the results can be critically compared with known results obtained by other methods.

A. Definition of the model and notations

Starting point of our calculation are the exact flow equations derived in Ref. 25, which give the RG flow of the one-particle irreducible vertices when degrees of freedom are integrated out and a subsequent rescaling step is applied. For convenience we shall now briefly present

these equations for the special case of $1d$, using a slightly different notation than in Ref. 25. We shall work at zero temperature, but expect that our results also describe temperatures $T > 0$ as long as we consider energy scales large compared with T. In $1d$, the Fermi surface consists of two points, αk_F, where $\alpha = \pm 1$ labels the right and left Fermi point. An arbitrary wave-vector k can be written as

$$k = \alpha(k_F + p) \ , \tag{2.1}$$

where $\alpha = +1$ for $k > 0$ and $\alpha = -1$ for $k < 0$. Note that the deviation $p = \alpha(k - \alpha k_F)$ from $\pm k_F$ is always measured locally outwards, which is different from the convention usually adopted in bosonization.[1,6,7,12] For small p we may expand the non-interacting energy dispersion ϵ_k around the Fermi points,

$$\epsilon_{\alpha(k_F+p)} = \epsilon_{k_F} + v_F p + \frac{p^2}{2m} + \dots \ . \tag{2.2}$$

Here m is the bare mass of the fermions. As discussed in Ref. 25, the expansion (2.2) is around the true Fermi momentum of the interacting many-body system. Of course, in $1d$ the interacting and non-interacting k_F are identical if we compare systems with the same density. Note that by time reversal invariance $\epsilon_k = \epsilon_{-k}$, so that v_F and m are independent of α. For $|p| \ll k_F$ it is reasonable to linearize the energy dispersion and neglect the term of order p^2 in Eq. (2.2). Introducing an infrared cutoff Λ and an ultraviolet cutoff Λ_0 with units of energy, the non-interacting Matsubara Green function is

$$G^0_{\Lambda,\Lambda_0}(\alpha(k_F+p), i\omega) = \frac{\Theta(\Lambda_0 > v_F|p| > \Lambda)}{i\omega - v_F p} \ , \tag{2.3}$$

where $i\omega$ is a fermionic Matsubara frequency, and

$$\begin{aligned} \Theta(x_2 > x > x_1) &= \Theta(x_2 - x) - \Theta(x_1 - x) \\ &= \begin{cases} 1 & \text{if } x_2 > x > x_1 \\ 0 & \text{else} \end{cases} \ . \end{aligned} \tag{2.4}$$

Later we shall also write

$$\Theta(x_2 > x_1) = \Theta(x_2 - x_1) = \begin{cases} 1 & \text{if } x_2 > x_1 \\ 0 & \text{else} \end{cases} \ . \tag{2.5}$$

Taking the limit $\Lambda_0 \to \infty$ in Eq. (2.3) amounts to extending the linear energy dispersion on both branches from $-\infty$ to $+\infty$. The unphysical states with energies far below the Fermi energy introduced in this way are occupied according to the Pauli principle. It is generally accepted that this filled Dirac sea is dynamically irrelevant and does not modify the low-energy physics. However, the precise way in which the cutoff is removed can affect the numerical value of the various Luttinger liquid parameters.[27] At this point we shall keep Λ_0 finite and assume that the bare electron-electron interaction of the model is characterized by a momentum- and frequency-independent totally antisymmetric irreducible four-point vertex of the form

$$\Gamma^{(4)}_{\Lambda_0}(K_1', K_2'; K_2, K_1) = A_{\alpha_1'\alpha_2';\alpha_2\alpha_1} g_0 \ , \tag{2.6}$$

where $K = (k, i\omega) = (\alpha(k_F + p), i\omega)$, and

$$\begin{aligned} A_{\alpha_1'\alpha_2';\alpha_2\alpha_1} &= \delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2} - \delta_{\alpha_2',\alpha_1}\delta_{\alpha_1',\alpha_2} \\ &= D_{\alpha_1'\alpha_2';\alpha_2\alpha_1} - E_{\alpha_1'\alpha_2';\alpha_2\alpha_1} \end{aligned} \tag{2.7}$$

is antisymmetric with respect to the exchange $\alpha_1 \leftrightarrow \alpha_2$ or $\alpha_1' \leftrightarrow \alpha_2'$. For later convenience we have introduced the notations

$$D_{\alpha_1'\alpha_2';\alpha_2\alpha_1} = \delta_{\alpha_1',\alpha_1}\delta_{\alpha_2',\alpha_2} \ , \ \text{(direct term)} \tag{2.8}$$

$$E_{\alpha_1'\alpha_2';\alpha_2\alpha_1} = \delta_{\alpha_2',\alpha_1}\delta_{\alpha_1',\alpha_2} \ , \ \text{(exchange term)} \ . \tag{2.9}$$

In the limits $\Lambda_0 \to \infty$ and $\Lambda \to 0$ the Green function of the model defined by Eqs. (2.3) and (2.6) can be calculated exactly in the space-time domain via bosonization (TLM with g_2-interactions). The generally accepted result for the spectral function is given in Eq. (1.2).[28]

B. Scaling functions and flow equations

Suppose now that we reduce the infrared cutoff by setting

$$\Lambda = \Lambda_0 e^{-l} \tag{2.10}$$

and follow the evolution of the correlation functions as the logarithmic flow parameter l increases. Noting that the infrared cutoff Λ has units of energy, we may define a length scale ξ and an associated time scale τ,

$$\xi = v_F/\Lambda \ \ , \ \ \tau = 1/\Lambda = \xi/v_F \ . \tag{2.11}$$

From these quantities, we may construct dimensionless scaling variables,[29]

$$q = p\xi \ = \ (\alpha k - k_F)\xi \ \ , \ \ \epsilon = \omega\tau = \omega/\Lambda \ , \tag{2.12}$$

and write the exact single-particle Matsubara Green function of the theory with infrared cutoff Λ and ultraviolet cutoff Λ_0 in the dynamic scaling form[13,14]

$$G_{\Lambda,\Lambda_0}(k, i\omega) = \tau Z_l \tilde{G}_l(p\xi, i\omega\tau) \ , \tag{2.13}$$

where $\tilde{G}_l(q, i\epsilon)$ is a dimensionless dynamic scaling function. Here the wave-function renormalization factor Z_l is related to the irreducible self-energy $\Sigma_\Lambda(k, i\omega)$ as usual,

$$Z_l = \frac{1}{1 - \left.\frac{\partial \Sigma_\Lambda(k_F, i\omega)}{\partial(i\omega)}\right|_{\omega=0}} \ . \tag{2.14}$$

Note that by time-reversal symmetry Z_l and Σ_Λ are independent of α. It is also convenient to set

$$\tilde{G}_l(q, i\epsilon) \ = \ \frac{\Theta(e^l > |q| > 1)}{r_l(q, i\epsilon)} \ , \tag{2.15}$$

where

$$r_l(q, i\epsilon) \ = \ Z_l\left[i\epsilon - q\right] + \tilde{\Gamma}^{(2)}_l(q, i\epsilon) \ , \tag{2.16}$$

and

$$\tilde{\Gamma}_l^{(2)}(q,i\epsilon) = -\tau Z_l \left[\Sigma_\Lambda(k_F + q/\xi, i\epsilon/\tau) - \Sigma(k_F, i0)\right] . \tag{2.17}$$

Here $\Sigma(k,i\omega) = \lim_{\Lambda\to 0}\Sigma_\Lambda(k,i\omega)$ is the exact self-energy of the model without infrared cutoff. Following Ref. 25, we also define the scaling functions for the higher order irreducible vertices,

$$\begin{aligned}\tilde{\Gamma}_l^{(2n)}(Q_1',\ldots,Q_n';Q_n,\ldots,Q_1) &= \\ \nu_0^{n-1}\Lambda^{n-2}Z_l^n\Gamma_\Lambda^{(2n)}(K_1',\ldots,K_n';K_n,\ldots,K_1)&,\end{aligned} \tag{2.18}$$

where $\Gamma_\Lambda^{(2n)}(K_1',\ldots,K_n';K_n,\ldots,K_1)$ are the usual one-particle irreducible $2n$-point vertices, $\nu_0 = (\pi v_F)^{-1}$ is the density of states of non-interacting spinless electrons in $1d$, and $K=(k,i\omega)$ and $Q=(\alpha,q,i\epsilon)$ are composite labels.

The rescaled irreducible two-point vertex defined in Eq. (2.17) satisfies the following exact flow equation,[25]

$$\partial_l\tilde{\Gamma}_l^{(2)}(Q) = (1-\eta_l - Q\cdot\partial_Q)\tilde{\Gamma}_l^{(2)}(Q) + \dot{\Gamma}_l^{(2)}(Q) , \tag{2.19}$$

where

$$\begin{aligned}\dot{\Gamma}_l^{(2)}(Q) &= -\frac{1}{2}\sum_{\alpha'}\int dq'\int\frac{d\epsilon'}{2\pi}\dot{G}_l(Q') \\ &\times \tilde{\Gamma}_l^{(4)}(Q,Q';Q',Q) ,\end{aligned} \tag{2.20}$$

and η_l is the flowing anomalous dimension,

$$\eta_l = -\partial_l \ln Z_l = -\frac{\partial_l Z_l}{Z_l} . \tag{2.21}$$

We have introduced the notations

$$Q\cdot\partial_Q = q\partial_q + \epsilon\partial_\epsilon , \tag{2.22}$$

$$\dot{G}_l(Q) = \frac{\delta(|q|-1)}{r_l(Q)} . \tag{2.23}$$

Eq. (2.19) is equivalent with the following integral equation,

$$\begin{aligned}\tilde{\Gamma}_l^{(2)}(Q) &= \tilde{\Gamma}_{l=0}^{(2)}(Q) \\ &+ \int_0^l dt\, e^{t-\bar{\eta}_l(t)}(1-\eta_{l-t})\tilde{\Gamma}_{l=0}^{(2)}(e^{-t}Q) \\ &+ \int_0^l dt\, e^{t-\bar{\eta}_l(t)}\dot{\Gamma}_{l-t}^{(2)}(e^{-t}Q) ,\end{aligned} \tag{2.24}$$

where $e^{-t}Q = (\alpha, e^{-t}q, e^{-t}i\epsilon)$, and we have defined

$$\bar{\eta}_l(t) = \int_0^t dt'\eta_{l-t+t'} . \tag{2.25}$$

The flow of the inhomogeneity $\dot{\Gamma}_l^{(2)}(Q)$ on the right-hand side of Eq. (2.19) is controlled by the scaling function of the four-point vertex $\tilde{\Gamma}_l^{(4)}$, which satisfies the exact flow equation

$$\partial_l\tilde{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) = -\left[2\eta_l + \sum_{i=1}^{2}\left(Q_i'\cdot\partial_{Q_i'} + Q_i\cdot\partial_{Q_i}\right)\right]\tilde{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) + \dot{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) , \tag{2.26}$$

where

$$\begin{aligned}\dot{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) &= -\frac{1}{2}\sum_\alpha\int dq\int\frac{d\epsilon}{2\pi}\dot{G}_l(Q)\tilde{\Gamma}_l^{(6)}(Q_1',Q_2',Q;Q,Q_2,Q_1) \\ &- \frac{1}{2}\sum_\alpha\int dq\int\frac{d\epsilon}{2\pi}\left[\dot{G}_l(Q)\tilde{G}_l(Q') + \tilde{G}_l(Q)\dot{G}_l(Q')\right] \\ &\times\left\{\frac{1}{2}\left[\tilde{\Gamma}_l^{(4)}(Q_1',Q_2';Q',Q)\tilde{\Gamma}_l^{(4)}(Q,Q';Q_2,Q_1)\right]_{K'=K_1+K_2-K}\right. \\ &- \left[\tilde{\Gamma}_l^{(4)}(Q_1',Q';Q,Q_1)\tilde{\Gamma}_l^{(4)}(Q_2',Q;Q',Q_2)\right]_{K'=K+K_1-K_1'} \\ &+\left.\left[\tilde{\Gamma}_l^{(4)}(Q_2',Q';Q,Q_1)\tilde{\Gamma}_l^{(4)}(Q_1',Q;Q',Q_2)\right]_{K'=K+K_1-K_2'}\right\} .\end{aligned} \tag{2.27}$$

Here $\tilde{\Gamma}_l^{(6)}$ is the dimensionless irreducible six-point vertex. As Eq. (2.20) can be converted into Eq. (2.24), Eq. (2.26)

can be transformed into the integral equation

$$\tilde{\Gamma}^{(4)}_l(Q'_1, Q'_2; Q_2, Q_1) = \tilde{\Gamma}^{(4)}_{l=0}(Q'_1, Q'_2; Q_2, Q_1) - \int_0^l dt e^{-2\bar{\eta}_l(t)} 2\eta_{l-t} \tilde{\Gamma}^{(4)}_{l=0}(e^{-t}Q'_1, e^{-t}Q'_2; e^{-t}Q_2, e^{-t}Q_1) + \int_0^l dt e^{-2\bar{\eta}_l(t)} \dot{\Gamma}^{(4)}_{l-t}(e^{-t}Q'_1, e^{-t}Q'_2; e^{-t}Q_2, e^{-t}Q_1) . \quad (2.28)$$

C. Classification of couplings in the hydrodynamic regime

The above functional RG equations determine the RG flow of the momentum- and frequency-dependent irreducible vertices $\tilde{\Gamma}^{(2n)}_l(Q'_1, \ldots, Q'_n; Q_n, \ldots, Q_1)$. By expanding these vertices in powers of the dimensionless scaling variables $q_i = p_i\xi$ and $\epsilon_i = \omega_i\tau$, we obtain the RG flow of the coupling constants of the model, which are the coefficients in this multi-dimensional Taylor expansion. Following the usual terminology, couplings with positive scaling dimension are called relevant, couplings with vanishing scaling dimension are called marginal, and couplings with negative scaling dimension are called irrelevant. The irrelevant couplings grow at short distances (i.e. in the *ultraviolet*) and spoil the renormalizabiliy of the theory. In this case the usual field theory RG is not applicable. On the other hand, in a Wilsonian RG the irrelevant couplings can be treated on the same footing as the relevant ones. For this reason the Wilsonian RG is very popular in condensed matter physics, where one usually has a physical ultraviolet cutoff and renormalizability is not necessary.

What is the regime of validity of the above expansion? Obviously, the expansion of the rescaled vertex functions to some finite order in powers of the dimensionless scaling variables $q_i = p_i\xi$ and $\epsilon_i = \omega_i\tau$ can be justified if these variables are small compared with unity. As long as the infrared cutoff $\Lambda = v_F/\xi = 1/\tau$ is finite, this condition can be satisfied for sufficiently small wave-vectors and frequencies. If we identify ξ with the correlation length and τ with the corresponding characteristic time scale, then the expansion of the $\tilde{\Gamma}^{(2n)}_l(Q'_1, \ldots, Q'_n; Q_n, \ldots, Q_1)$ in powers of the scaling variables $q_i = p_i\xi$ and $\epsilon_i = \omega_i\tau$ can be viewed as an expansion of the dynamic scaling functions in the *hydrodynamic regime*, where one is interested in length scales larger than ξ and time scales longer than τ. Of course, a normal metal at zero temperature is a critical system, where ξ and τ diverge. Close to this critical state (where ξ and τ are large but finite), we expect that the correlation functions assume some scaling form. This corresponds to the *critical regime* (also called *scaling regime*). For example, we may describe a normal metal at finite temperature in terms of flow equations derived for $T = 0$ if we stop scaling the flow equations at a finite length scale of the order of $\xi \approx v_F/T$.

For certain systems it may happen that the couplings defined in the hydrodynamic regime smoothly evolve into analogous couplings in the scaling regime, where both q_i and ϵ_i are large. This is the essence of the dynamic scaling hypothesis.[13] For example, for a Fermi liquid we expect that such a smooth crossover between the hydrodynamic and the scaling regime indeed exists, because by definition the self-energy is analytic and hence can be expanded in powers of momenta and frequencies. On the other hand, the two-point function of a Luttinger liquid is known to exhibit algebraic singularities, so that there cannot be a smooth connection between the hydrodynamic and the critical regime. In this case one expects the analytic properties of the dynamic scaling function $\tilde{\Gamma}^{(2)}_l(q, i\epsilon)$ to change as we move from the hydrodynamic into the scaling regime. In this work we shall show that this is indeed the case and give an approximate expression for the scaling function.

Let us now classify the couplings according to their relevance in the hydrodynamic regime. First of all, the momentum- and frequency-independent part of the two-point vertex,

$$\tilde{\mu}_l = \tilde{\Gamma}^{(2)}_l(0) , \quad (2.29)$$

is relevant. Here $\tilde{\Gamma}^{(2)}_l(0)$ stands for $\tilde{\Gamma}^{(2)}_l(\alpha, q = 0, i\epsilon = i0)$. From Eq. (2.19) it is easy to see that $\tilde{\mu}_l$ satisfies the exact flow equation

$$\partial_l \tilde{\mu}_l = (1 - \eta_l)\tilde{\mu}_l + \dot{\Gamma}^{(2)}_l(0) . \quad (2.30)$$

Note that in general we have to fine-tune the other couplings such that the relevant coupling $\tilde{\mu}_l$ approaches a fixed point. The fixed point value of $\tilde{\mu}_l$ for $l \to \infty$ is then determined by

$$(1 - \eta_\infty)\tilde{\mu}_\infty = -\dot{\Gamma}^{(2)}_\infty(0) . \quad (2.31)$$

Because $\dot{\Gamma}^{(2)}_\infty(0)$ implicitly depends on $\tilde{\mu}_\infty$, Eq. (2.31) is really a self-consistency equation for the fixed point value of $\tilde{\mu}_l$.

There are two marginal couplings associated with the irreducible two-point vertex $\tilde{\Gamma}^{(2)}_l(Q)$: the wave-function renormalization factor Z_l, and the Fermi velocity renormalization factor $\tilde{v}_l$. Using the definition (2.14), it is easy to show that Z_l can be expressed in terms of our rescaled two-point vertex as follows,

$$Z_l = 1 - \left. \frac{\partial \tilde{\Gamma}^{(2)}_l(\alpha, 0, i\epsilon)}{\partial(i\epsilon)} \right|_{\epsilon=0} . \quad (2.32)$$

The dimensionless Fermi velocity renormalization factor can be written as

$$\tilde{v}_l = Z_l - \left. \frac{\partial \tilde{\Gamma}_l^{(2)}(\alpha, q, i0)}{\partial q} \right|_{q=0} . \tag{2.33}$$

Note that for a Fermi liquid $\tilde{v}_l$ can be identified with the dimensionless inverse effective mass renormalization,

$$\tilde{v}_l = \frac{m}{m_l^*} , \tag{2.34}$$

where m is the bare mass and m_l^* is the effective mass of the model with infrared cutoff $\Lambda_0 e^{-l}$. The expansion of the scaling function associated with the two-point vertex for small q and ϵ then reads

$$\begin{aligned} \tilde{\Gamma}_l^{(2)}(Q) &= \tilde{\mu}_l + (1 - Z_l) i\epsilon + (Z_l - \tilde{v}_l) q \\ &\quad + \mathcal{O}(q^2, \epsilon^2, q\epsilon) , \end{aligned} \tag{2.35}$$

so that for small q and ϵ the dimensionless inverse propagator defined in Eq. (2.16) is given by

$$\begin{aligned} r_l(Q) &= Z_l \left[i\epsilon - q \right] + \tilde{\Gamma}_l^{(2)}(Q) \\ &= i\epsilon - \tilde{v}_l q + \tilde{\mu}_l + \mathcal{O}(q^2, \epsilon^2, q\epsilon) . \end{aligned} \tag{2.36}$$

The exact flow equations for the marginal couplings Z_l and $\tilde{v}_l$ are easily obtained from Eqs. (2.19), (2.21) and (2.33),

$$\partial_l Z_l = -\eta_l Z_l , \tag{2.37}$$

$$\partial_l \tilde{v}_l = -\eta_l \tilde{v}_l - \left. \frac{\partial \dot{\Gamma}_l^{(2)}(\alpha, q, i0)}{\partial q} \right|_{q=0} . \tag{2.38}$$

Differentiating Eq. (2.19) with respect to $i\epsilon$, setting $\epsilon = 0$, and using Eq. (2.32), we see that the flowing anomalous dimension η_l is related to the function $\dot{\Gamma}_l^{(2)}(Q)$ via

$$\eta_l = \left. \frac{\partial \dot{\Gamma}_l^{(2)}(\alpha, 0, i\epsilon)}{\partial (i\epsilon)} \right|_{\epsilon=0} . \tag{2.39}$$

Since $\dot{\Gamma}_l^{(2)}(\alpha, 0, i\epsilon)$ is defined in terms of the four-point vertex $\tilde{\Gamma}_l^{(4)}$ via Eq. (2.20), we can obtain the following explicit expression for the flowing anomalous dimension,

$$\begin{aligned} \eta_l &= -\frac{1}{2} \sum_{\alpha'} \int dq' \int \frac{d\epsilon'}{2\pi} \dot{G}_l(Q') \\ &\quad \times \left. \frac{\partial \tilde{\Gamma}_l^{(4)}(\alpha, 0, i\epsilon, Q'; Q', \alpha, 0, i\epsilon)}{\partial (i\epsilon)} \right|_{\epsilon=0} . \end{aligned} \tag{2.40}$$

This expression is also valid in dimensions $d > 1$ provided the discrete index α is replaced by a d-dimensional unit vector which labels the points on the Fermi surface. Thus, for a given irreducible four-point vertex, Eq. (2.40) can be used to directly calculate the anomalous dimension of any normal Fermi system.

There is one more marginal coupling, namely the irreducible four-point vertex at vanishing momenta and frequencies. Because by construction $\tilde{\Gamma}_l^{(4)}(Q_1', Q_2'; Q_2, Q_1)$ is antisymmetric with respect to the permutation of the incoming or the outgoing fermions, for vanishing external momenta and frequencies the four-point vertex must be of the form

$$\tilde{\Gamma}_l^{(4)}(\alpha_1', 0, \alpha_2', 0; \alpha_2, 0, \alpha_1, 0) = A_{\alpha_1' \alpha_2'; \alpha_2 \alpha_1} \tilde{g}_l , \tag{2.41}$$

where $A_{\alpha_1' \alpha_2'; \alpha_2 \alpha_1}$ is defined in Eq. (2.7). Hence, the marginal part of the four-point vertex can be expressed in terms of a single marginal coupling $\tilde{g}_l$, which satisfies the exact flow equation

$$\partial_l \tilde{g}_l = -2\eta_l \tilde{g}_l + B_l . \tag{2.42}$$

Here B_l is defined by

$$\dot{\Gamma}_l^{(4)}(\alpha_1', 0, \alpha_2', 0; \alpha_2, 0, \alpha_1, 0) = A_{\alpha_1' \alpha_2'; \alpha_2 \alpha_1} B_l , \tag{2.43}$$

with $\dot{\Gamma}_l^{(4)}$ given in Eq. (2.27).

It is convenient to subtract the relevant and marginal couplings from the exact vertices, thus defining coupling functions containing only irrelevant couplings. The irrelevant part of the two-point vertex is

$$\begin{aligned} \tilde{\Gamma}_l^{(2-\mu,Z,v)}(Q) &= \tilde{\Gamma}_l^{(2)}(Q) - \tilde{\Gamma}_l^{(2)}(\alpha, 0, i0) \\ &\quad - i\epsilon \left. \frac{\partial \tilde{\Gamma}_l^{(2)}(\alpha, 0, i\epsilon)}{\partial i\epsilon} \right|_{\epsilon=0} - q \left. \frac{\partial \tilde{\Gamma}_l^{(2)}(\alpha, q, i0)}{\partial q} \right|_{q=0} \\ &= \tilde{\Gamma}_l^{(2)}(Q) - \tilde{\mu}_l - i\epsilon(1 - Z_l) - q(Z_l - \tilde{v}_l) , \end{aligned} \tag{2.44}$$

where the superscripts indicate the marginal and relevant couplings to be subtracted. The rescaled inverse propagator $r_l(q, i\epsilon)$ defined in Eq. (2.16) can then be written as

$$r_l(q, i\epsilon) = i\epsilon - \tilde{v}_l q + \tilde{\mu}_l + \tilde{\Gamma}_l^{(2-\mu,Z,v)}(Q) . \tag{2.45}$$

Finally, the irrelevant part of the four-point vertex is

$$\begin{aligned} &\tilde{\Gamma}_l^{(4-g)}(Q_1', Q_2'; Q_2, Q_1) \\ &\quad = \tilde{\Gamma}_l^{(4)}(Q_1', Q_2'; Q_2, Q_1) - \left. \tilde{\Gamma}_l^{(4)}(Q_1', Q_2'; Q_2, Q_1) \right|_{q_i = \epsilon_i = 0} \\ &\quad = \tilde{\Gamma}_l^{(4)}(Q_1', Q_2'; Q_2, Q_1) - A_{\alpha_1' \alpha_2'; \alpha_2 \alpha_1} \tilde{g}_l . \end{aligned} \tag{2.46}$$

The exact flow equations for the subtracted vertices are formally identical with the flow equations for the unsubtracted vertices given in Eqs. (2.19) and (2.26), except for the fact that the inhomogeneities $\dot{\Gamma}_l^{(2)}(Q)$ and $\dot{\Gamma}_l^{(4)}(Q_1', Q_2'; Q_2, Q_1)$ on the right-hand sides should be replaced by their subtracted versions,

$$\begin{aligned} \dot{\Gamma}_l^{(2-\mu,Z,v)}(Q) &= \dot{\Gamma}_l^{(2)}(Q) - \dot{\Gamma}_l^{(2)}(\alpha, 0, i0) \\ &\quad - i\epsilon \left. \frac{\partial \dot{\Gamma}_l^{(2)}(\alpha, 0, i\epsilon)}{\partial i\epsilon} \right|_{\epsilon=0} - q \left. \frac{\partial \dot{\Gamma}_l^{(2)}(\alpha, q, i0)}{\partial q} \right|_{q=0} , \end{aligned} \tag{2.47}$$

$$\dot{\Gamma}_l^{(4-g)}(Q_1',Q_2';Q_2,Q_1) = \dot{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) - \dot{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1)\Big|_{q_i=\epsilon_i=0} . \quad (2.48)$$

III. APPROXIMATE SOLUTION OF THE FUNCTIONAL FLOW EQUATIONS FOR THE TWO-POINT VERTEX

So far no approximation has been made, except for the linearization of the energy dispersion. In order to make progress, we need to truncate the hierarchy of flow equations. We now give a truncation scheme which at weak coupling reproduces the known scaling properties of the spectral function $A(\pm(k_F+p),\omega)$ of the TLM. Moreover, at least for $p=0$, we recover the exact weak coupling result for the spectral function known from bosonization.

A. One-loop flow

To begin with, let us briefly discuss the flow of the relevant and the marginal couplings within the one-loop approximation. We first consider the two-point vertex. The exact flow equation is given in Eq. (2.19). The flow is coupled to the irreducible four-point vertex via the inhomogeneity $\dot{\Gamma}_l^{(2)}(Q)$ on the right-hand side, as given in Eq. (2.20). Within the one-loop approximation, it is sufficient to approximate the four-point vertex appearing in Eq. (2.20) by

$$\tilde{\Gamma}_l^{(4)}(Q,Q';Q',Q) = \delta_{\alpha,-\alpha'}\tilde{g}_l + \mathcal{O}(\tilde{g}_l^2) , \quad (3.1)$$

where we have used the fact that $A_{\alpha\alpha';\alpha'\alpha} = 1-\delta_{\alpha,\alpha'} = \delta_{\alpha,-\alpha'}$. In this approximation

$$\begin{aligned}\dot{\Gamma}_l^{(2)}(Q) &= -\frac{\tilde{g}_l}{2}\sum_{\alpha'}\int dq'\int\frac{d\epsilon'}{2\pi}\,\delta_{\alpha,-\alpha'}\dot{G}_l(Q') + \mathcal{O}(\tilde{g}_l^2)\\ &= -\frac{\tilde{g}_l}{2}\left[\Theta(\tilde{\mu}_l+\tilde{v}_l)+\Theta(\tilde{\mu}_l-\tilde{v}_l)\right] + \mathcal{O}(\tilde{g}_l^2) . \quad (3.2)\end{aligned}$$

For $|\tilde{\mu}_l| < |\tilde{v}_l|$ this simplifies to

$$\dot{\Gamma}_l^{(2)}(Q) = -\frac{\tilde{g}_l}{2} + \mathcal{O}(\tilde{g}_l^2) \ , \ \ |\tilde{\mu}_l| < |\tilde{v}_l| \ . \quad (3.3)$$

As shown below, in the physically relevant case $\tilde{\mu}_l = \mathcal{O}(\tilde{g}_l)$, while $\tilde{v}_l$ is of the order of unity. From Eqs. (2.30) and (2.37)-(2.39) we conclude that for small $\tilde{g}_l$

$$\begin{aligned}\partial_l\tilde{\mu}_l &= \tilde{\mu}_l - \frac{\tilde{g}_l}{2} , & (3.4)\\ \partial_l Z_l &= 0 , & (3.5)\\ \partial_l\tilde{v}_l &= 0 . & (3.6)\end{aligned}$$

Finally, let us consider the momentum- and frequency-independent part of the four-point vertex. Using the fact that $\tilde{\Gamma}_l^{(6)}$ is of order $\tilde{g}_l^3$ and hence can be neglected,[18,30] we obtain from Eq. (2.27) to second order in $\tilde{g}_l$,

$$\begin{aligned}\dot{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) \approx& -\frac{\tilde{g}_l^2}{2}\sum_{\alpha}\int dq\int\frac{d\epsilon}{2\pi}\\ &\times\Big\{\frac{1}{2}\big[A_{\alpha_1'\alpha_2';\alpha'\alpha}A_{\alpha\alpha';\alpha_2\alpha_1}\\ &\quad\times\Big(\dot{G}_l(Q)\tilde{G}_l(Q')+\tilde{G}_l(Q)\dot{G}_l(Q')\Big)\big]_{K'=K_1+K_2-K}\\ &-\big[A_{\alpha_1'\alpha';\alpha\alpha_1}A_{\alpha_2'\alpha;\alpha'\alpha_2}\\ &\quad\times\Big(\dot{G}_l(Q)\tilde{G}_l(Q')+\tilde{G}_l(Q)\dot{G}_l(Q')\Big)\big]_{K'=K+K_1-K_1'}\\ &+\big[A_{\alpha_2'\alpha';\alpha\alpha_1}A_{\alpha_1'\alpha;\alpha'\alpha_2}\\ &\quad\times\Big(\dot{G}_l(Q)\tilde{G}_l(Q')+\tilde{G}_l(Q)\dot{G}_l(Q')\Big)\big]_{K'=K+K_1-K_2'}\Big\} . \quad (3.7)\end{aligned}$$

Because on the critical manifold $\tilde{\mu}_l = \mathcal{O}(\tilde{g}_l)$, we may approximate on the right-hand side of Eq. (3.7)

$$\tilde{G}_l(Q) \approx \frac{\Theta(e^l > |q| > 1)}{i\epsilon - \tilde{v}_l q} \ , \ \ \dot{G}_l(Q) \approx \frac{\delta(|q|-1)}{i\epsilon - \tilde{v}_l q} . \quad (3.8)$$

The integrations in Eq. (3.7) can then be performed. We obtain for the final result

$$\begin{aligned}&\dot{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) \approx\\ &-\tilde{g}_l^2\Big\{A_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\dot{\chi}_l(q_1-q_2,i\epsilon_1+i\epsilon_2)\\ &+E_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\big[\delta_{\alpha_1,\alpha_2}\dot{\Pi}_l(q_1-q_1',i\epsilon_1-i\epsilon_1')\\ &\qquad+\delta_{\alpha_1,-\alpha_2}\dot{\chi}_l(q_1+q_1',i\epsilon_1-i\epsilon_1')\big]\\ &-D_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\big[\delta_{\alpha_1,\alpha_2}\dot{\Pi}_l(q_1-q_2',i\epsilon_1-i\epsilon_2')\\ &\qquad+\delta_{\alpha_1,-\alpha_2}\dot{\chi}_l(q_1+q_2',i\epsilon_1-i\epsilon_2')\big]\Big\} . \quad (3.9)\end{aligned}$$

Here

$$\dot{\chi}_l(q,i\epsilon) = \Theta(e^l-1>|q|)\frac{\tilde{v}_l(2+|q|)}{\epsilon^2+\tilde{v}_l^2(2+|q|)^2} , \quad (3.10)$$

$$\dot{\Pi}_l(q,i\epsilon) = \Theta(e^l+1>|q|>2)\frac{s_q}{i\epsilon+\tilde{v}_l q} , \quad (3.11)$$

where we have introduced the notation

$$s_q = \mathrm{sign}(q) . \quad (3.12)$$

Setting external momenta and frequencies equal to zero we obtain from Eq. (3.9)

$$\begin{aligned}\dot{\Gamma}_l^{(4)}(\alpha_1',0,\alpha_2',0;\alpha_2,0,\alpha_1,0) &= -\tilde{g}_l^2 A_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\\ &\times\Big\{\delta_{\alpha_1,\alpha_2}\dot{\chi}_l(0,i0)-\delta_{\alpha_1,\alpha_2}\dot{\Pi}_l(0,i0)\Big\} = 0 . \quad (3.13)\end{aligned}$$

Hence, the coefficient of order $\tilde{g}_l^2$ in the weak coupling expansion of the function B_l defined in Eq. (2.43) vanishes. Anticipating that $\eta_l = \mathcal{O}(\tilde{g}_l^2)$, we obtain from Eq. (2.42) within the one-loop approximation

$$\partial_l\tilde{g}_l = 0 . \quad (3.14)$$

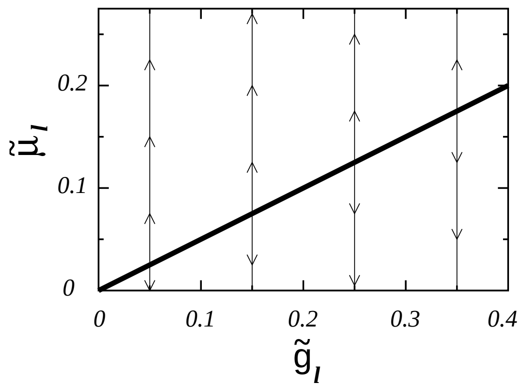

FIG. 1: One-loop RG flow in the $\tilde{g}-\tilde{\mu}$-plane, see Eqs. (3.4) and (3.14). The thick solid line is a line of fixed points.

The vanishing of the β-function of the marginal coupling $\tilde{g}_l$ of the TLM is well-known. However, the one-loop RG flow of the relevant coupling $\tilde{\mu}_l$ is non-trivial, because the initial value of $\tilde{\mu}_l$ has to be fine-tuned such that $\tilde{\mu}_l$ does not exhibit a runaway flow. As emphasized in Ref. 25, the requirement that $\tilde{\mu}_l$ flows into a fixed point is equivalent with the statement that the initial k_F is the correct Fermi momentum, in agreement with the Luttinger theorem.[26,31] The flow in the $\tilde{g}-\tilde{\mu}$-plane implied by Eqs. (3.4) and (3.14) is shown in Fig. 1. Obviously, at the one-loop level the marginal couplings $\tilde{g}_l$, Z_l and $\tilde{v}_l$ do not flow, while the relevant coupling $\tilde{\mu}_l$ exhibits a runaway flow unless the initial value is chosen such that

$$\tilde{\mu}_0 = \frac{\tilde{g}_0}{2} \ . \tag{3.15}$$

In this case $\tilde{\mu}_l = \frac{\tilde{g}_0}{2}$ for all l, so that we obtain a RG fixed point. The one-loop flow equations given above have also been discussed by Shankar,[32] who derived these equations within the framework of the conventional momentum shell technique. Because within the one-loop approximation $Z_l = 1$, this approximation is not sufficient to detect the non-Fermi liquid behavior of our model.

B. Two-loop flow of the two-point vertex

It is now straightforward to calculate the full two-point vertex within the two-loop approximation and to study the emergence of Luttinger liquid behavior when the infrared cutoff is reduced. The crucial observation is that for large values of the flow parameter l the momentum- and frequency-dependent part of the four-point vertex $\tilde{\Gamma}_l^{(4-g)}(Q_1',Q_2';Q_2,Q_1)$ can be expanded in powers of the marginal coupling $\tilde{g}_l$. This follows from the fact that the RG trajectory approaches the manifold defined by the relevant and marginal couplings, so that the irrelevant couplings become local functions of the relevant and marginal ones.[18,30]

To calculate the two-point vertex at the two-loop order, we need to know the momentum- and frequency-dependent part of the four-point vertex within the one-loop approximation. By definition,

$$\tilde{\Gamma}_l^{(4)}(Q_1',Q_2';Q_2,Q_1) = A_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\tilde{g}_l + \tilde{\Gamma}_l^{(4-g)}(Q_1',Q_2';Q_2,Q_1) \ , \tag{3.16}$$

where to order $\tilde{g}_l^2$ we obtain from Eqs. (2.28) and (3.9),

$$\begin{aligned}\tilde{\Gamma}_l^{(4-g)}(Q_1',Q_2';Q_2,Q_1) \ &\approx \ \tilde{\Gamma}_{l=0}^{(4-g)}(Q_1',Q_2';Q_2,Q_1) - \tilde{g}_l^2\Big\{A_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\chi_l(q_1-q_2,i\epsilon_1+i\epsilon_2) \\ &+ E_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\left[\delta_{\alpha_1,\alpha_2}\Pi_l(q_1-q_1',i\epsilon_1-i\epsilon_1') + \delta_{\alpha_1,-\alpha_2}\chi_l(q_1+q_1',i\epsilon_1-i\epsilon_1')\right] \\ &- D_{\alpha_1'\alpha_2';\alpha_2\alpha_1}\left[\delta_{\alpha_1,\alpha_2}\Pi_l(q_1-q_2',i\epsilon_1-i\epsilon_2') + \delta_{\alpha_1,-\alpha_2}\chi_l(q_1+q_2',i\epsilon_1-i\epsilon_2')\right]\Big\} \ . \end{aligned} \tag{3.17}$$

Here

$$\Pi_l(q,i\epsilon) = \Pi_{l,0}(q,i\epsilon) \ \ , \ \ \chi_l(q,i\epsilon) = \chi_{l,0}(q,i\epsilon) \ , \tag{3.18}$$

where for arbitrary initial value $l_0 \geq 0$,

$$\begin{aligned}\chi_{l,l_0}(q,i\epsilon) \ &= \ \int_{l_0}^{l} dt \left[\dot{\chi}_{l-t}(e^{-t}q,e^{-t}i\epsilon) - \dot{\chi}_{l-t}(0,i0)\right] \\ &\approx \ -\frac{1}{2\tilde{v}_l}\left\{(l-l_0)\Theta(|q|>e^l-e^{l_0}) + \frac{1}{2}\ln\left[\frac{\tilde{v}_l^2(2+e^{-l_0}|q|)^2+e^{-2l_0}\epsilon^2}{\tilde{v}_l^2(2-e^{-l}|q|)^2+e^{-2l}\epsilon^2}\right]\Theta(e^l-e^{l_0}>|q|)\right\} \ , \end{aligned} \tag{3.19}$$

and

$$\Pi_{l,l_0}(q,i\epsilon) = \int_{l_0}^{l} dt \left[\dot{\Pi}_{l-t}(e^{-t}q, e^{-t}i\epsilon) - \dot{\Pi}_{l-t}(0,i0)\right]$$
$$\approx \frac{1}{2}\,\frac{s_q}{i\epsilon + \tilde{v}_l q}\left\{(|q| - 2e^{l_0})\Theta(e^l + e^{l_0} > |q| > 2e^{l_0}) + (2e^l - |q|)\Theta(2e^l > |q| > e^l + e^{l_0})\right\} . \quad (3.20)$$

Note that Eq. (3.17) is manifestly antisymmetric with respect to the exchange of the incoming or the outgoing labels. For the calculation of the two-point vertex we only need the irreducible four-point vertex at $Q_1 = Q_1'$ and $Q_2 = Q_2'$, i.e.

$$\tilde{\Gamma}_l^{(4)}(Q,Q';Q',Q) = \delta_{\alpha,-\alpha'}\tilde{g}_l + \delta_{\alpha,\alpha'}\tilde{g}_l^2 \Pi_l(q-q', i\epsilon - i\epsilon')$$
$$+\delta_{\alpha,-\alpha'}\tilde{g}_l^2\left[\chi_l(q+q', i\epsilon - i\epsilon') + \chi_l(q-q', i\epsilon + i\epsilon')\right] . \quad (3.21)$$

A graphical representation of this equation is shown in Fig. 2. We recognize the usual perturbative contributions to the effective interaction: the first term of order $\tilde{g}_l^2$ on the right-hand side of Eq. (3.21) corresponds to the contribution from the zero-sound (ZS) channel, the second term corresponds to the Peierls channel (sometimes also denoted by ZS′ channel), and the last term corresponds to the BCS channel. Note that the zero-sound channel gives rise to an effective retarded forward scattering interaction of the g_4-type (using the g-ology language), which is generated by integrating out degrees of freedom.

Given an approximate expression for the four-point vertex, we may calculate the inhomogeneity $\dot{\Gamma}_l^{(2)}(Q)$ on the right-hand side of the flow equation for the two-point vertex, see Eqs. (2.19) and (2.20). Separating the contributions due to the zero sound (ZS) and the Peierls-BCS (PB) channels, we have

$$\dot{\Gamma}_l^{(2)}(q,i\epsilon) = -\frac{\tilde{g}_l}{2} + \dot{\Gamma}_l^{(2,\mathrm{ZS})}(q,i\epsilon) + \dot{\Gamma}_l^{(2,\mathrm{PB})}(q,i\epsilon) + \mathcal{O}(\tilde{g}_l^3) , \quad (3.22)$$

where

$$\dot{\Gamma}_l^{(2,\mathrm{ZS})}(q,i\epsilon)$$
$$= -\frac{\tilde{g}_l^2}{2}\int dq' \int \frac{d\epsilon'}{2\pi}\frac{\delta(|q'|-1)}{i\epsilon' - \tilde{v}_l q'}\,\Pi_l(q-q', i\epsilon - i\epsilon')$$
$$= -\frac{\tilde{g}_l^2}{4} s_q\Big[\Theta(e^l > |q| > 1)\,\frac{|q|-1}{\tilde{v}_l(2+|q|) + i\epsilon s_q}$$
$$+ \Theta(2e^l - 1 > |q| > e^l)\,\frac{2e^l - 1 - |q|}{\tilde{v}_l(2+|q|) + i\epsilon s_q}\Big] , \quad (3.23)$$

$$\dot{\Gamma}_l^{(2,\mathrm{PB})}(q,i\epsilon) = -\frac{\tilde{g}_l^2}{2}\int dq' \int \frac{d\epsilon'}{2\pi}\frac{\delta(|q'|-1)}{i\epsilon' - \tilde{v}_l q'}$$
$$\times\left[\chi_l(q+q', i\epsilon - i\epsilon') - \chi_l(q-q', i\epsilon + i\epsilon')\right]$$
$$= \frac{\tilde{g}_l^2}{4\tilde{v}_l} s_q\Bigg\{\Theta(1 > |q| > 2 - e^l)\ln\left[\frac{\tilde{v}_l(4-|q|) + i\epsilon s_q}{\tilde{v}_l(2e^l + |q|) + i\epsilon s_q}\right]$$
$$+ \Theta(e^l > |q| > 1)\ln\left[\frac{\tilde{v}_l(2+|q|) + i\epsilon s_q}{\tilde{v}_l(2e^l + 2 - |q|) + i\epsilon s_q}\right]$$
$$- \Theta(e^l - 2 > |q|)\ln\left[\frac{\tilde{v}_l(4+|q|) - i\epsilon s_q}{\tilde{v}_l(2e^l - |q|) - i\epsilon s_q}\right]\Bigg\} . \quad (3.24)$$

Expanding to first order in q and ϵ, we obtain

$$\dot{\Gamma}_l^{(2,\mathrm{PB})}(q,i\epsilon) = \frac{\tilde{g}_l^2}{8\tilde{v}_l^2}\Theta(e^l - 2)\Big[(1 - 2e^{-l})i\epsilon$$
$$-(1 + 2e^{-l})\tilde{v}_l q + \mathcal{O}(\epsilon^2, q^2, \epsilon q)\Big] . \quad (3.25)$$

Because $\dot{\Gamma}_l^{(2,\mathrm{ZS})}(q,i\epsilon)$ vanishes for $|q| < 1$, it does not contribute to the flow of the marginal couplings Z_l and $\tilde{v}_l$. From Eq. (2.39) we obtain for the anomalous dimension at scale l,

$$\eta_l = \frac{\tilde{g}_l^2}{8\tilde{v}_l^2}\Theta(e^l - 2)(1 - 2e^{-l}) , \quad (3.26)$$

and from Eq. (2.38) we find for the flow of the Fermi velocity renormalization factor

$$\partial_l \tilde{v}_l = \frac{\tilde{g}_l^2}{2\tilde{v}_l}\Theta(e^l - 2)e^{-l} . \quad (3.27)$$

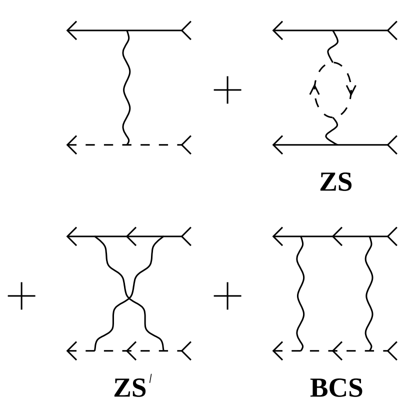

FIG. 2: Feynman diagrams contributing to the effective interaction within the one-loop approximation, see Eq. (3.21). The solid (dashed) arrows represent the fermion propagators with momenta close to the left (right) Fermi point, and the wavy lines represent the flowing coupling constant $\tilde{g}_l$.

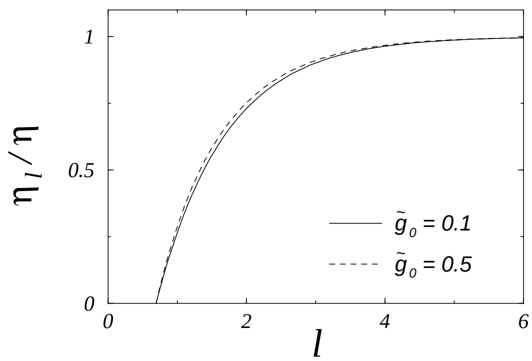

FIG. 3: Anomalous dimension η_l as a function of the logarithmic flow parameter l, see Eqs. (3.26) and (3.27).

Using the fact that $\tilde{g}_l = \tilde{g}$ is independent of the flow parameter l, Eq. (3.27) can be easily integrated,

$$\tilde{v}_l = \tilde{v}_0 \Big[1 + \Theta(e^l - 2) \frac{\tilde{g}^2}{2\tilde{v}_0^2}(1 - 2e^{-l}) \Big]^{1/2} . \tag{3.28}$$

Note that $\tilde{v}_l$ rapidly approaches a constant $\tilde{v}_\infty$ for large l, which is to leading order in $\tilde{g}$ given by

$$\tilde{v}_\infty = \tilde{v}_0 + \frac{\tilde{g}^2}{4\tilde{v}_0} + \mathcal{O}(\tilde{g}^4) . \tag{3.29}$$

In Fig. 3 we show the flow of the anomalous dimension η_l given in Eq. (3.26) as a function of the logarithmic flow parameter l. Obviously, η_l vanishes for $l < \ln 2$, and approaches a constant η for large l, which is given by

$$\eta = \frac{\tilde{g}^2}{8\tilde{v}_\infty} . \tag{3.30}$$

The above fixed point value of η agrees with the weak coupling expansion of the bosonization result for the anomalous dimension of the spinless g_2-TLM. Note, however, that the exact solubility of the TLM relies on the filled Dirac sea associated with the linearized energy dispersion. As discussed by Schulz and Shastry,[27] the introduction of the Dirac sea leads to finite renormalizations of the fixed point values of the Luttinger liquid parameters (such as the anomalous dimension), so that the value of η beyond the leading order $\tilde{g}$ is modified. This can be explicitly verified within our RG approach, where the introduction of the Dirac sea corresponds to the removal of the ultraviolet cutoff Λ_0. Then we should take the limit $e^l \to \infty$ in Eqs. (3.23) and (3.24), so that the explicit l-dependence on the right-hand sides of these expressions disappears. In this case the marginal coupling $\tilde{v}_l$ is not renormalized at all ($\partial_l \tilde{v}_l = 0$), so that $\tilde{v}_\infty = \tilde{v}_0$. Furthermore, the initial flow of the anomalous dimension η_l discussed above is also removed by the Dirac sea, so that η_l is effectively replaced by its asymptotic limit η given in Eq. (3.30). To simplify the following calculation of the spectral function, we shall from now on work with a Dirac sea, which amounts to taking the limits $\Lambda_0 \to \infty$ and $l \to \infty$ in Eqs. (3.23) and (3.24). Then we obtain

$$\begin{aligned} \dot{\Gamma}_\infty^{(2,\mathrm{ZS})}(q, i\epsilon) &= -\Theta(|q| > 1) 2\eta s_q \frac{|q| - 1}{\tilde{v}_0(2 + |q|) + i\epsilon s_q} \\ &= \Theta(|q| > 1)\eta s_q \left[1 - \frac{3\tilde{v}_0|q| + i\epsilon s_q}{\tilde{v}_0(2 + |q|) + i\epsilon s_q} \right] , \end{aligned} \tag{3.31}$$

$$\begin{aligned} \dot{\Gamma}_\infty^{(2,\mathrm{PB})}(q, i\epsilon) = 2\eta s_q \Big\{ &\Theta(|q| > 1) \ln[\tilde{v}_0(2 + |q|) + i\epsilon s_q] \\ &+ \Theta(1 > |q|) \ln[\tilde{v}_0(4 - |q|) + i\epsilon s_q] \\ &- \ln[\tilde{v}_0(4 + |q|) - i\epsilon s_q] \Big\} . \end{aligned} \tag{3.32}$$

The corresponding subtracted function $\dot{\Gamma}_\infty^{(2-\mu,Z,v)}(Q)$ defined in Eq. (2.47) is in this approximation given by

$$\begin{aligned} \dot{\Gamma}_\infty^{(2-\mu,Z,v)}(q, i\epsilon) = \eta s_q \Big\{ &2\Theta(|q| > 1) \ln[\tilde{v}_0(2 + |q|) + i\epsilon s_q] \\ &+ 2\Theta(1 > |q|) \ln[\tilde{v}_0(4 - |q|) + i\epsilon s_q] \\ &- 2\ln[\tilde{v}_0(4 + |q|) - i\epsilon s_q] \\ &+ \Theta(|q| > 1) \left[1 - \frac{3\tilde{v}_0|q| + i\epsilon s_q}{\tilde{v}_0(2 + |q|) + i\epsilon s_q} \right] \\ &- (i\epsilon s_q - \tilde{v}_0|q|) \Big\} . \end{aligned} \tag{3.33}$$

So far we have calculated the function $\dot{\Gamma}_\infty^{(2-\mu,Z,v)}(Q)$ defined in Eq. (2.47) perturbatively to second order in $\tilde{g}$. Let us for the moment proceed further within perturbation theory. In this case we may set $\eta_l = 0$ on the right-hand side of the exact flow equation (2.19) for the two-point vertex, which amounts to setting $\bar{\eta}_l(t) = 0$ in Eq. (2.24). Assuming that initially $\tilde{\Gamma}_{l=0}^{(2-\mu,Z,v)}(Q) = 0$, we obtain in this approximation

$$\begin{aligned} \tilde{\Gamma}_l^{(2-\mu,Z,v)}(q, i\epsilon) &\approx \textstyle\int_0^l dt\, e^t \dot{\Gamma}_\infty^{(2-\mu,Z,v)}(e^{-t}q, e^{-t}i\epsilon) \\ &= \textstyle\int_{e^{-l}}^1 d\lambda\, \lambda^{-2} \dot{\Gamma}_\infty^{(2-\mu,Z,v)}(\lambda q, \lambda i\epsilon) . \end{aligned} \tag{3.34}$$

Substituting the perturbative result (3.33) into Eq. (3.34), going back to unrescaled variables $p = \Lambda q/v_F$, $\omega = \Lambda\epsilon$, $\tilde{\Gamma}_l^{(2)}(Q) = -\frac{Z_l}{\Lambda}[\Sigma_\Lambda(k, i\omega) - \Sigma(\alpha k_F, i0)]$ (see Eq. (2.17)), and finally taking the scaling limit $l \to \infty$ (i.e. $\Lambda \to 0$), we recover the well-known perturbative self-energy of the spinless g_2-TLM,

$$\begin{aligned} \Sigma(\alpha k_F + \alpha p, i\omega) - \Sigma(\alpha k_F, i\omega) \approx \\ -\frac{\tilde{g}^2}{8} v_F p + \frac{\tilde{g}^2}{16}(i\omega - v_F p) \ln\left[\frac{(v_F p)^2 + \omega^2}{\xi_0^2} \right] . \end{aligned} \tag{3.35}$$

It is also easy to check that in the regime

$$1 \ll |q| \ll e^l \ , \ 1 \ll |\epsilon| \ll e^l , \tag{3.36}$$

we may approximate

$$\tilde{\Gamma}_l^{(2-\mu,Z,v)}(q, i\epsilon) \approx \tilde{\Gamma}_{l\to\infty}^{(2-\mu,Z,v)}(q, i\epsilon) , \tag{3.37}$$

so that the only l-dependence of the self-energy enters via the scaling variables $q = p\xi = p\xi_0 e^l$ and $\epsilon = \omega\tau = \omega/\Lambda = \omega e^l/\Lambda_0$. Thus, in the scaling limit the spectral function can be written in terms of a scaling function which does not explicitly depend on the logarithmic scale factor l.

We now propose a simple procedure to go beyond perturbation theory: from Eq. (2.24) we know that the exact two-point vertex for $l \gg 1$ satisfies

$$\begin{aligned}\tilde{\Gamma}_l^{(2-\mu,Z,v)}(q,i\epsilon) &= \int_0^l dt\, e^{(1-\eta)t}\, \dot{\Gamma}_\infty^{(2-\mu,Z,v)}(e^{-t}q, e^{-t}i\epsilon) \\ &= \int_{e^{-l}}^1 d\lambda\, \lambda^{-2+\eta}\, \dot{\Gamma}_\infty^{(2-\mu,Z,v)}(\lambda q, \lambda i\epsilon)\ , \qquad (3.38)\end{aligned}$$

which differs from the perturbative expression (3.34) because the fixed-point value of the anomalous dimension appears on the right-hand side. We now approximate the flow function $\dot{\Gamma}_\infty^{(2-\mu,Z,v)}(q,i\epsilon)$ on the right-hand side of Eq. (3.38) by

$$\begin{aligned}\dot{\Gamma}_\infty^{(2-\mu,Z,v)}(q,i\epsilon) \approx \eta s_q \Big\{ & 2\Theta(|q|>1)\ln[\tilde{v}(2+|q|) + i\epsilon s_q] \\ & + 2\Theta(1>|q|)\ln[\tilde{v}(4-|q|) + i\epsilon s_q] \\ & - 2\ln[\tilde{v}(4+|q|) - i\epsilon s_q] \\ & - \Theta(|q|>1)\frac{3\tilde{v}|q| + i\epsilon s_q}{\tilde{v}(2+|q|) + i\epsilon s_q} \\ & - (i\epsilon s_q - \tilde{v}|q|)\Big\}\ . \qquad (3.39)\end{aligned}$$

This expression is almost identical with the perturbative two-loop result (3.33), except that the term $\eta s_q \Theta(|q|>1)$ has been omitted, and the factor $\tilde{v}_0$ has been replaced by the perturbative Fermi velocity renormalization factor

$$\tilde{v} = 1 - \frac{\tilde{g}^2}{8} = 1 - \eta\ , \qquad (3.40)$$

see Eq. (1.4). Note that within perturbation theory the term $\eta s_q \Theta(|q|>1)$ in Eq. (3.33) is responsible for the finite Fermi velocity renormalization given by the first term $-\frac{\tilde{g}^2}{8} v_F p$ on the right-hand side of Eq. (3.35). By substituting $\tilde{v}_0 \to \tilde{v}$ in Eq. (3.33), we have implicitly replaced the bare propagator in the scaling limit by $[i\epsilon - \tilde{v}q]^{-1}$, thus taking the Fermi velocity renormalization in the scaling limit (where the dimensionless variables p and ϵ are large compared with unity) self-consistently into account. Note that according to Eq. (2.34) the dimensionless quantity $\tilde{v}$ can be identified with the renormalization of the inverse effective mass, which remains finite in $1d$.

Given Eqs. (3.38) and (3.39), we may calculate the complete dynamic scaling function for the irreducible self-energy, describing the change in scaling behavior as we move from the hydrodynamic into the scaling regime. In the hydrodynamic regime the scaling function is analytic in q and ϵ. On the other hand, for large $|q|$ and $|\epsilon|$ the scaling function exhibits algebraic singularities corresponding to Luttinger liquid behavior. In the scaling regime (3.36) we may use the approximation (3.37) and obtain from Eqs. (3.38) and (3.39)

$$\begin{aligned}\tilde{\Gamma}_\infty^{(2-\mu,Z,v)}(q,i\epsilon) &= \frac{\eta}{2}(i\epsilon - \tilde{v}q) \\ &\times\Bigg\{ f\left(\frac{i\epsilon s_q - \tilde{v}|q|}{4\tilde{v}}\right) + f\left(-\frac{i\epsilon s_q + \tilde{v}|q|}{2\tilde{v}}\right) \\ &+ |q|^{-\eta}\left[f\left(-\frac{i\epsilon s_q - \tilde{v}|q|}{4\tilde{v}|q|}\right) - f\left(-\frac{i\epsilon s_q + \tilde{v}|q|}{2\tilde{v}|q|}\right)\right]\Bigg\}\ , \qquad (3.41)\end{aligned}$$

where the complex function $f(z)$ is defined by

$$f(z) = z\int_0^1 d\lambda\, \frac{\lambda^\eta}{1 - z\lambda} = z\ {}_2F_1(1, 1+\eta, 2+\eta; z)\ . \qquad (3.42)$$

Here, ${}_2F_1(a,b,c;z)$ is a hypergeometric function. Note that Eq. (3.41) is restricted to the scaling regime defined by (3.36). To obtain the spectral function, we analytically continue $\tilde{\Gamma}_\infty^{(2-\mu,Z,v)}(q,i\epsilon)$ according to $i\epsilon \to \epsilon + i0$. Using

$$\mathrm{Im}\, f(x \pm i0) = \pm\pi\Theta(x>1)x^{-\eta}\ , \qquad (3.43)$$

and assuming that $|\epsilon^2 - \tilde{v}^2q^2| \gg 1$, we obtain

$$\begin{aligned}\mathrm{Im}\,\tilde{\Gamma}_\infty^{(2-\mu,Z,v)}(q,\epsilon+i0) &= 2\pi\eta \\ &\times\Bigg\{\Theta(\epsilon s_q > \tilde{v}|q|)\left|\frac{\epsilon - \tilde{v}q}{4\tilde{v}}\right|^{1-\eta} \\ &+ \Theta(-\tilde{v}|q| > \epsilon s_q > -3\tilde{v}|q|)\left|\frac{\epsilon - \tilde{v}q}{4\tilde{v}}\right|\left|\frac{\epsilon + \tilde{v}q}{2\tilde{v}}\right|^{-\eta} \\ &+ \Theta(-3\tilde{v}|q| > \epsilon s_q)\left|\frac{\epsilon - \tilde{v}q}{4\tilde{v}}\right|^{1-\eta}\Bigg\}\ . \qquad (3.44)\end{aligned}$$

Mathematically the imaginary part of $\tilde{\Gamma}_\infty^{(2-\mu,Z,v)}(q,\epsilon+i0)$ arises from the branch cut of $f(x+iy)$ on the real axis for $x > 1$. Furthermore, using the fact that for $|\arg(-z)| < \pi$,

$$f(z) = \frac{\pi(1+\eta)}{\sin(\pi\eta)}(-z)^{-\eta} - \frac{1+\eta}{\eta}\,{}_2F_1(1,-\eta,1-\eta;\frac{1}{z})\ , \qquad (3.45)$$

we obtain in the regime $|\epsilon^2 - \tilde{v}^2q^2| \gg 1$ and $\eta \ll 1$ for the real part,

$$\begin{aligned}\mathrm{Re}\,\tilde{\Gamma}_\infty^{(2-\mu,Z,v)}(q,\epsilon+i0) &= -(\epsilon - \tilde{v}q) \\ &+ \Theta(\epsilon s_q > -3\tilde{v}|q|)(\epsilon - \tilde{v}q)\frac{1}{2}\left[\left|\frac{\epsilon - \tilde{v}q}{4\tilde{v}}\right|^{-\eta} + \left|\frac{\epsilon + \tilde{v}q}{2\tilde{v}}\right|^{-\eta}\right] \\ &+ \Theta(-3\tilde{v}|q| > \epsilon s_q)(\epsilon - \tilde{v}q)\left|\frac{\epsilon - \tilde{v}q}{4\tilde{v}}\right|^{-\eta}\ . \qquad (3.46)\end{aligned}$$

In deriving this expression, we have neglected corrections of the order of η to the prefactor. Note that the imaginary part given in Eq. (3.44) is proportional to η, while no

such small prefactor appears in the real part, Eq. (3.46). The first term on the right-hand side of Eq. (3.46) exactly cancels the inverse bare propagator $i\epsilon - \tilde{v}q$ such that $r_\infty(q, \epsilon + i0) = (\epsilon - \tilde{v}q) + \tilde{\Gamma}_\infty^{(2-\mu,Z,v)}(q, \epsilon + i0)$ has the dynamic scaling property[13]

$$r_\infty(q, \epsilon + i0) = |q|^{1-\eta} \tilde{f}_\infty^{\pm}(\epsilon/\tilde{v}q) , \qquad (3.47)$$

where the functions $f_\infty^+(x)$ and $f_\infty^-(x)$ refer to $q > 0$ and $q < 0$ respectively and $(f_\infty^+(x))^* = -f_\infty^-(x)$. A graph of $\tilde{f}_\infty^+(\epsilon/\tilde{v}q)$ is shown in Fig. 4 for $\eta = 0.2$.

Using the fact that for $\xi \gg \xi_0$ we may identify

$$Z_l = Z_0\, e^{-\int_0^l dt\, \eta_t} \equiv \bar{Z}_0\, e^{-\eta l} \approx \left(\frac{\xi_0}{\xi}\right)^\eta , \qquad (3.48)$$

where the constant $\bar{Z}_0 = \mathcal{O}(1)$ has been absorbed by ξ_0, the corresponding spectral function can be written in the scaling form similar to Eq. (1.9),

$$A(\alpha(k_F + p), \omega) = \tau \left(\frac{\xi_0}{\xi}\right)^\eta \tilde{A}_\infty(p\xi, \omega\tau) , \qquad (3.49)$$

where $\tau = \xi/v_F = 1/\Lambda$, and the scaling function $\tilde{A}_\infty(q, \epsilon)$ is

$$\begin{aligned} \tilde{A}_\infty(q,\epsilon) &= -\frac{1}{\pi}\,\mathrm{Im}\left[\frac{1}{\epsilon - \tilde{v}q + \tilde{\mu}_\infty + \tilde{\Gamma}_\infty^{(2-\mu,Z,v)}(q,\epsilon+i0)}\right] = \frac{\eta}{2}|\epsilon - \tilde{v}q|^{\eta-1} \\ &\times \begin{cases} 4\left[1 + \left|\frac{\epsilon-\tilde{v}q}{\epsilon+\tilde{v}q}\right|^\eta\right]^{-2} & \text{for } \epsilon s_q > \tilde{v}|q| \\ 0 & \text{for } \tilde{v}|q| > \epsilon s_q > -\tilde{v}|q| \\ 4\left[\left|2\frac{\epsilon+\tilde{v}q}{\epsilon-\tilde{v}q}\right|^{\eta/2} + \left|\frac{1}{2}\frac{\epsilon-\tilde{v}q}{\epsilon+\tilde{v}q}\right|^{\eta/2}\right]^{-2} & \text{for } -\tilde{v}|q| > \epsilon s_q > -3\tilde{v}|q| \\ 1 & \text{for } -3\tilde{v}|q| > \epsilon s_q \ . \end{cases} \end{aligned} \qquad (3.50)$$

In the prefactor we have retained only the leading order in η. For $q = 0$, the dynamic scaling function reduces to

$$\tilde{A}_\infty(0, \epsilon) = \frac{\eta}{2}|\epsilon|^{\eta-1} . \qquad (3.51)$$

If $q \neq 0$, the dynamic scaling function can be written in the dynamic scaling form[13]

$$\tilde{A}_\infty(q, \epsilon) = |q|^{\eta-1} \tilde{h}_\infty(\epsilon/\tilde{v}q) , \qquad (3.52)$$

where $\tilde{h}_\infty(x)$ is defined via Eq. (3.50). A plot of the dynamic scaling function $\tilde{h}_\infty(\epsilon/\tilde{v}q)$ is given in Fig. 5 for $\eta = 0.2$. For a comparison, we have also plotted the generally accepted scaling function of the TLM, given in Eq. (1.7). The qualitative agreement between the two plots becomes even better for smaller η. Recall that Eq. (3.50) is only valid for $|q| \gg 1$, $|\epsilon| \gg 1$ *and for large* $\big||\epsilon| - \tilde{v}|q|\big|$. In this limit the small constant $\tilde{\mu}_\infty$ on the right-hand side of Eq. (3.50) can be neglected. In terms of the physical variables $p = q/\xi$ and $\omega = \epsilon/\tau = \epsilon v_F/\xi$ the condition $\big||\epsilon| - \tilde{v}|q|\big| \gg 1$ becomes $\big||\omega| - v_c|p|\big| \gg 1/\xi$, where $v_c = \tilde{v}v_F$. Using the fact that the function $\tilde{A}_\infty(q, \epsilon)$ satisfies the scaling relation

$$\tilde{A}_\infty(sq, s\epsilon) = s^{\eta-1} \tilde{A}_\infty(q, \epsilon) \qquad (3.53)$$

we may write $\tilde{A}_\infty(p\xi, \omega\tau) = \tau^{\eta-1} \tilde{A}_\infty(v_F p, \omega)$. With $\tau(\xi_0/\xi)^\eta \tau^{\eta-1} = \Lambda_0^{-\eta}$ we finally obtain for the spectral function at the Luttinger liquid fixed point,

$$\begin{aligned} A(\alpha(k_F+p),\omega) &= \Lambda_0^{-\eta}\frac{\eta}{2}|\omega - v_c p|^{\eta-1} \\ &\times \begin{cases} 4\left[1 + \left|\frac{\omega-v_c p}{\omega+v_c p}\right|^\eta\right]^{-2} & \text{for } \omega s_p > v_c|p| \\ 0 & \text{for } v_c|p| > \omega s_p > -v_c|p| \\ 4\left[\left|2\frac{\omega+v_c p}{\omega-v_c p}\right|^{\eta/2} + \left|\frac{1}{2}\frac{\omega-v_c p}{\omega+v_c p}\right|^{\eta/2}\right]^{-2} & \text{for } -v_c|p| > \omega s_p > -3v_c|p| \\ 1 & \text{for } -3v_c|p| > \omega s_p \ . \end{cases} \end{aligned} \qquad (3.54)$$

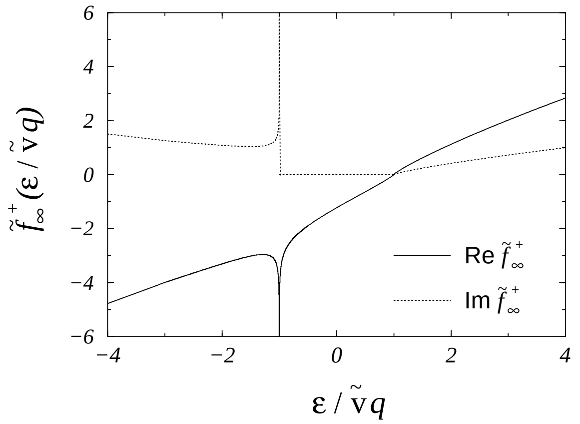

FIG. 4: Real and imaginary part of $\tilde{f}^{+}_{\infty}(\epsilon/\tilde{v}q) \equiv r_{\infty}(q,\epsilon + i0)/|q|^{1-\eta} = \left[(\epsilon - \tilde{v}q) + \tilde{\Gamma}^{(2-\mu,Z,v)}_{\infty}(q,\epsilon+i0)\right]/|q|^{1-\eta}$ for $\eta = 0.2$, where $\tilde{\Gamma}^{(2-\mu,Z,v)}_{\infty}(q,\epsilon+i0)$ is the subtracted irreducible two-point scaling function, see Eqs. (3.44) and (3.46).

Let us now compare Eq. (3.54) with the bosonization result for the TLM given in Eq. (1.2). First of all, for $p = 0$ we obtain from Eq. (3.54)

$$A(\alpha k_F, \omega) = \Lambda_0^{-\eta}\frac{\eta}{2}|\omega|^{\eta-1} \ , \tag{3.55}$$

which corresponds to Eq. (3.52) and agrees exactly with the result (1.2) obtained via bosonization.Thus, at least for $p = 0$, our truncation of the exact RG flow equations is reliable. Moreover, Eq. (3.54) exhibits the same scaling behavior $A(\alpha k_F + \alpha s p, s\omega) = s^{\eta-1}A(\alpha k_F + \alpha p, \omega)$ as the expression (1.2) obtained via bosonization. For finite p, however, the detailed line-shape of the spectral function predicted by Eq. (3.54) is different from Eq. (1.2). In particular, for $\omega \to v_c p + 0$ the bosonization result (1.2) predicts

$$A_{\rm TL}(k_F + p, \omega) \sim \Lambda_0^{-\eta}\frac{\eta}{2}|2v_c p|^{\eta/2}|\omega - v_c p|^{-1+\eta/2} \ , \tag{3.56}$$

while our RG calculation yields

$$A(k_F + p, \omega) \sim \Lambda_0^{-\eta} 2\eta|\omega - v_c p|^{-1+\eta} \ , \tag{3.57}$$

which diverges with a different exponent than Eq. (3.56). Similarly, for $\omega \to -v_c p - 0$ the bosonization result (1.2) predicts

$$A_{\rm TL}(k_F + p, \omega) \sim \Lambda_0^{-\eta}\frac{\eta}{2}|2v_c p|^{-1+\eta/2}|\omega + v_c p|^{\eta/2} \ , \tag{3.58}$$

while Eq. (3.54) yields

$$A(k_F + p, \omega) \sim \Lambda_0^{-\eta} 2\eta|2v_c p|^{-1}|\omega + v_c p|^{\eta} \ . \tag{3.59}$$

Note, however, that Eq. (3.57) has been derived assuming $|\epsilon^2 - \tilde{v}^2 q^2| \gg 1$, which means in terms of the physical variables $\big||\omega| - v_c|p|\big| \gg 1/\tau = \Lambda$. For $\Lambda \to 0$ this condition is always satisfied unless precisely $\omega = \pm v_c p$. This limiting procedure describes the spectral function precisely at the critical point, because the length $\xi = v_F/\Lambda$ associated with the infrared cutoff can be sent to infinity *before* the limit $\omega \to v_c p$ is taken. Obviously, there exists another regime where both $|\epsilon| = |\omega|\tau$ and $|q| = |p|\xi$ are large, but where

$$\big||\epsilon| - \tilde{v}|q|\big| \lesssim 1 \ . \tag{3.60}$$

This corresponds to first taking the limit $\omega \to \pm v_c p$, and then removing the infrared cutoff $\Lambda = 1/\tau \to 0$. In this limit the spectral function behaves completely different. Keeping in mind that in practice the TLM describes a physical system only above some finite energy scale Λ_* below which backscattering becomes important or a crossover to higher dimensionality sets in, this limiting procedure is physically relevant to describe quasi $1d$ systems. If we naively use our result for the irreducible self-energy given in Eq. (3.41), we find with the help of Eq. (3.45) that for $\epsilon \to \tilde{v}q$ and for small η

$$\tilde{\Gamma}^{(2-\mu,Z,v)}_{\infty}(q, i\epsilon) \approx \frac{|q|^{-\eta}}{2}(i\epsilon - \tilde{v}q) \ . \tag{3.61}$$

Since by assumption $|q| \gg 1$, this is a negligible correction to the bare inverse propagator $i\epsilon - \tilde{v}q$. It follows that in this regime our scaling function is simply

$$\tilde{A}_{\infty}(q,\epsilon) \approx \delta(\epsilon - \tilde{v}q) \ , \tag{3.62}$$

so that the physical spectral function exhibits a small but still finite quasiparticle peak with a weight of the order of $Z_{l_*} = (\Lambda_*/\Lambda_0)^{\eta}$. Note that this quasiparticle peak is due to the finite crossover energy scale Λ_* and does not occur in the TLM at zero temperature where

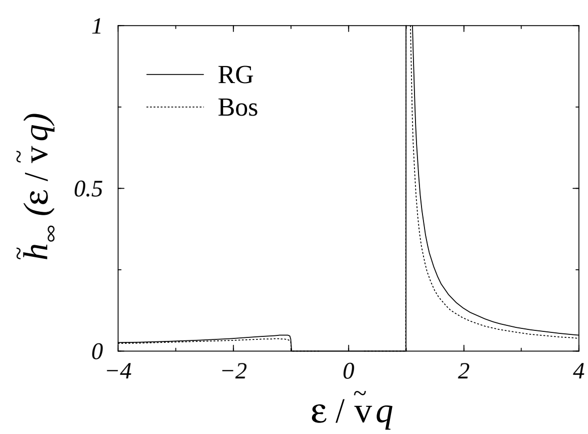

FIG. 5: Graph of the dynamic scaling function $\tilde{h}_{\infty}(\epsilon/\tilde{v}q) \equiv \tilde{A}_{\infty}(q,\epsilon)/|q|^{\eta-1}$ for $\eta = 0.2$, see Eqs. (3.49) and (3.50). For smaller η the RG result is indistinguishable from the bosonization result $\tilde{h}_{\rm TL}(\epsilon/\tilde{v}q)$ on this scale.

the infrared cutoff Λ can be reduced to zero. In deriving Eq. (3.62) we have introduced the Dirac sea, ignored the non-linearity in the energy dispersion as well as other irrelevant couplings, which will broaden the δ-function. In principle all these effects can be systematically taken into account within our formalism. The calculations are quite tedious and are beyond the scope of this work. It seems, however, that in the regime (3.60) the behavior of the dynamic scaling function of a generic quasi $1d$ interacting Fermi system with dominant forward scattering is not correctly described by the scaling function of the Tomonaga-Luttinger model $\tilde{A}_{\rm TL}(q,\epsilon)$ given in Eq. (1.6).

IV. SUMMARY AND CONCLUSIONS

In this work we have shown how to calculate the momentum- and frequency-dependent single-particle spectral function by means of a particular version of the functional RG method.[25] For simplicity, we have applied this method to the exactly solvable TLM. Our truncation scheme of the exact hierarchy of RG flow equations consists of the following steps:

1. Calculate the flow of the relevant and marginal couplings perturbatively for small interactions and adjust the initial values such that the RG flow approaches a fixed point.

2. Calculate the momentum- and frequency-dependent part of the four-point vertex (which involves an infinite number of irrelevant couplings) in powers of the marginal part of the four-point vertex.

3. Calculate the anomalous dimension in powers of the marginal part of the four-point vertex.

4. Substitute the results of steps 2. and 3. into the exact flow equation of the two-point vertex and then solve this equation exactly.

The result for the spectral function $A(k,\omega)$ is quite encouraging: it has the correct scaling properties, and for $k=\pm k_F$ agrees with the exact bosonization result. For finite $k-k_F$ and ω, we have found evidence that the spectral line shape in the vicinity of the Luttinger liquid fixed point exhibits some non-universal features.

Our work also shows how non-Fermi liquid behavior in a strongly correlated Fermi system can be detected using modern functional RG methods. Recently several authors have presented numerical studies of the one-loop flow equations for the marginal part of the *unrescaled* four-point vertex for Hubbard models in $2d$.[23,24] The marginal part is obtained by setting all frequencies equal to zero and projecting all wave-vectors onto the Fermi surface. Ignoring ambiguities[33] related to different orders of limits, the unrescaled part of the four-point vertex is then approximated by

$$\Gamma^{(4)}_{\Lambda}(\mathbf{k}'_1,\omega'_1,\mathbf{k}'_2,\omega'_2;\mathbf{k}_2,\omega_2,\mathbf{k}_1,\omega_1) \approx \Gamma^{(4)}_{\Lambda}(\mathbf{k}'_{F1},0,\mathbf{k}'_{F2},0;\mathbf{k}_{F2},0,\mathbf{k}_{F1},0)\ . \quad (4.1)$$

For the $2d$ Hubbard model with next-nearest neighbor hopping the numerical analysis[23,24] of the one-loop flow equations for the set of marginal couplings contained in the vertices $\Gamma^{(4)}_{\Lambda}(\mathbf{k}'_{F1},0,\mathbf{k}'_{F2},0;\mathbf{k}_{F2},0,\mathbf{k}_{F1},0)$ exhibits a runaway flow to strong coupling at a finite length scale $l_*=\ln(\Lambda_0/\Lambda_*)$, which is typically of the order of 10^2. The usual interpretation of this runaway flow is that it signals some instability of the normal metallic state.[23,24] In Ref. 25 we have pointed out that this runaway flow might also be an artifact of the one-loop approximation, and that the properly renormalized vertices could remain finite at the two-loop order. The crucial point is that the renormalized vertices involve wave-function renormalization factors (see Eq. (2.18)), which for a strongly correlated system can cancel a possibly strong enhancement of the unrescaled vertices. Our simple $1d$ model allows us to study such a scenario in detail. In this case $\Gamma^{(4)}_{\Lambda}$ can be parameterized in terms of a single marginal coupling g_l, which is defined in analogy to Eqs. (2.6) and (2.41),

$$\Gamma^{(4)}_{\Lambda_0 e^{-l}}(\alpha'_1 k_F,0,\alpha'_2 k_F,0;\alpha_2 k_F,0,\alpha_1 k_F,0) = A_{\alpha'_1\alpha'_2;\alpha_2\alpha_1} g_l\ . \quad (4.2)$$

At the two-loop order, the flow equation of g_l for our model is,[34]

$$\partial_l g_l = \frac{\nu_0^2}{4}\, g_l^3\ . \quad (4.3)$$

This implies

$$g_l = \frac{g_0}{\sqrt{1-\frac{\nu_0^2}{2}\, g_0^2 l}}\ . \quad (4.4)$$

Obviously, g_l diverges at a finite scale $l_*=2/(\nu_0 g_0)^2$. However, this divergence and the associated runaway-flow to strong coupling are unphysical, because the coupling g_l in Eq. (4.2) is *not* the properly renormalized coupling that can be identified with the usual g_2-interaction of the TLM. The latter is defined as a model describing the *fixed point* of the RG. Thus, the g_2-coupling that appears in the TLM should be identified with the *renormalized coupling at the RG fixed point*, which is related to the fixed point value of our rescaled coupling $\tilde{g}_l$,

$$\nu_0 g_2 = \lim_{l\to\infty}\tilde{g}_l = \lim_{l\to\infty}\left[Z_l^2 \nu_0 g_l\right]\ , \quad (4.5)$$

see Eqs. (2.18) and (2.41). Similar relations between vertex functions and interaction parameters at the RG fixed point are well known from Landau Fermi liquid theory.[35] In our simple model with a linearized energy dispersion the two-loop flow equation of the rescaled coupling is simply $\partial_l\tilde{g}_l=0$, so that the limit $l\to\infty$ in Eq. (4.5) indeed

exists. Note that the vanishing wave-function renormalization factor Z_l exactly compensates the diverging unrescaled coupling g_l such that the rescaled coupling $\tilde{g}_l$ remains finite and small. We believe that the above interpretation of the runaway flow of the RG-flow equations for the marginal part of the unrescaled vertices $\Gamma_\Lambda^{(4)}$ is not only specific to $1d$, where the RG fixed point does not correspond to a Fermi liquid. Assuming that in $2d$ the RG fixed point corresponds to a strongly correlated Fermi liquid with $Z_\infty \ll 1$, we expect that the unrescaled vertices given in Eq. (4.1) will flow to a finite but large value, which numerically might be indistinguishable from a runaway flow to infinity. At the same time, the properly rescaled vertices $\tilde{\Gamma}_l^{(4)}$ defined in Eq. (2.18) can remain finite and small.

The problem of calculating correlation functions of many-body systems at finite wave-vectors or frequencies using RG methods has not received much attention. For classical systems, functional RG calculations of the momentum-dependence of correlation functions can be found in the textbook by Ivanchenko and Lisyansky.[36] Here we have presented a functional RG calculation of a momentum- and frequency-dependent correlation function of a non-trivial quantum-mechanical many-body system. For a special $2d$ system a similar calculation has recently been performed by Ferraz,[16] who used the field theory RG. We believe that the method described in this work will also be useful to study other problems where no exact solutions are available. For example, the spectral function of one-dimensional Fermi systems where backscattering or Umklapp scattering are relevant cannot be calculated exactly by means of bosonization or other methods. Using our RG method, it should be possible to obtain the spectral function of Luther-Emery liquids even away from the Luther-Emery point.

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